

Higher dimensional inhomogeneous dust collapse and cosmic censorship

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Abstract

We investigate the occurrence and nature of a naked singularity in the gravitational collapse of an inhomogeneous dust cloud described by higher dimensional Tolman-Bondi space-times. The naked singularities are found to be gravitationally strong in the sense of Tipler. Higher dimensions seem to favour black holes rather than naked singularities.

Key Words: Naked singularity, cosmic censorship, higher dimensions

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1 Introduction

The work on inhomogeneous spherical collapse was pioneered in 1934 by Tolman [1] who found the metric for space-time with dust which was extended by Bondi [2]. Since then the metric, known as the Tolman-Bondi metric, is extensively used to study the formation of naked singularities in spherical collapse. It is seen that the Tolman-Bondi metric admits both naked and covered singularities

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depending upon the choice of initial data and that there is a smooth transition from one phase to the other [3]-[10]. However, according to the cosmic censorship conjecture (CCC) [11], the singularities that appear in gravitational collapse are always surrounded by an event horizon. Moreover, according to the strong version of the CCC, such singularities are not even locally naked, i.e., no non-spacelike curve can emerge from such singularities (see [12] for reviews on the CCC). The CCC has as yet no precise mathematical formulation or proof for either version. Hence the CCC remains one of the most important unsolved problems in general relativity and gravitation theory today. Consequently examples which appear to violate the CCC are important and they are an important tool to study this important issue.

Motivated by the development of superstring and other field theories, there is a renewed interest towards models with extra dimensions from the viewpoint of both cosmology [13] and gravitational collapse [14]. In this context, one question which is important and naturally arises is, would the examples of naked singularities in four-dimensional (4D) spherical gravitational collapse go over to higher dimensional (HD) space-time or not? If the answer to this question is yes, then a related question is whether the dimensionality of space-time has any effect on the formation and nature of the singularity. It is therefore interesting to study gravitational collapse of inhomogeneous dust in HD space-times. There is some literature on HD inhomogeneous dust collapse [15] from the viewpoint of CCC. However, these studies are restricted to 5D space-time and cannot reduce to the 4D case.

In this paper we generalize previous studies of 4D spherical inhomogeneous gravitational collapse to $(n+2)$ -dimensional space-times (where $n \geq 2$) and show that gravitational collapse of higher-dimensional space-times gives rise to shell focusing naked singularities which are also gravitationally strong. The conventional picture of spherical gravitational collapse described by 4D Tolman-Bondi space-time can be recovered from our analysis. This would be discussed in section III, which will be followed by our concluding remarks. In section II we review

HD Tolman-Bondi solutions.

The metric considered here is the generalization of the Tolman-Bondi metrics. We have used units which fix the speed of light and the gravitational constant via $8\pi G = c^4 = 1$.

2 Higher Dimensional Tolman-Bondi Solution

The HD Tolman-Bondi metric is given in co-moving coordinates as:

$$ds^2 = dt^2 - e^{\lambda(r,t)} dr^2 - R^2(t, r) d\Omega^2 \quad (1)$$

where

$$d\Omega^2 = \sum_{i=1}^{n+1} \left[\prod_{j=1}^{i-1} \sin^2(\theta_j) \right] d\theta_i^2 = d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2(d\theta_3^2 + \dots + \sin^2\theta_{n-1}d\theta_n^2)) \quad (2)$$

is the metric on an n -sphere and $n = D - 2$ (where D is the total number of dimensions), together with the stress-energy tensor for dust:

$$T_{ab} = \epsilon(t, r) \delta_a^t \delta_b^t \quad (3)$$

where $u_a = \delta_t^a$ is the $(n + 2)$ -dimensional velocity. The coordinate r is the co-moving radial coordinate, t is the proper time of freely falling shells, R is a function of t and r with $R > 0$. With the metric (1), the independent non-vanishing Einstein tensor components are

$$G_t^t = \frac{n(n-1)}{2R^2} (e^{-\lambda} R'^2 - \dot{R}^2 - 1) - \frac{n}{2} \frac{1}{R} (\dot{R}\dot{\lambda} + e^{-\lambda} R' \lambda') + n e^{-\lambda} \frac{R''}{R} = \epsilon \quad (4)$$

$$G_r^r = \frac{n(n-1)}{2R^2} (e^{-\lambda} R'^2 - \dot{R}^2 - 1) - n \frac{\ddot{R}}{R} = 0 \quad (5)$$

$$G_{\theta_1}^{\theta_1} = G_{\theta_2}^{\theta_2} = \dots = G_{\theta_n}^{\theta_n} = \frac{(n-1)(n-2)}{2} \frac{1}{R^2} (e^{-\lambda} R'^2 - \dot{R}^2 - 1) - \frac{n-1}{2} \frac{1}{R} (\dot{R}\dot{\lambda} + e^{-\lambda} R' \lambda') - (n-1) \frac{1}{R} (\ddot{R} - e^{-\lambda} R'') - \frac{1}{2} (\ddot{\lambda} + \frac{\lambda^2}{2}) = 0 \quad (6)$$

$$G_r^t = \frac{n}{2R}(2\dot{R}' - \lambda R') = 0 \quad (7)$$

where an over-dot and prime denote the partial derivative with respect to t and r , respectively.

Integration of Eq. (7) gives

$$e^{\lambda(r,t)} = \frac{R'^2}{1 + f(r)} \quad (8)$$

which can be substituted into Eq. (5) to yield

$$\dot{R}^2 = \frac{F(r)}{R^{(n-1)}} + f(r) \quad (9)$$

The functions $F(r)$ and $f(r)$ are arbitrary and are referred as the mass and energy functions, respectively. Since in the present discussion we are concerned with gravitational collapse, we require that $\dot{R}(t, r) < 0$. The energy density $\epsilon(t, r)$ is calculated as

$$\epsilon(t, r) = \frac{nF'}{2R^n R'} \quad (10)$$

For physical reasons, one assumes that the energy density ϵ is everywhere nonnegative. The special case $f(r) = 0$ corresponds to the marginally bound case which is of interest to us in this paper. In this case Eq. (9) can easily be integrated to

$$t - t_c(r) = -\frac{2}{(n+1)} \frac{R^{(n+1)/2}}{\sqrt{F}} \quad (11)$$

where $t_c(r)$ is an arbitrary function of integration which represents the proper time for the complete collapse of a shell with coordinate r . As it is possible to make an arbitrary re-labelling of spherical dust shells by $r \rightarrow g(r)$, without loss of generality, we fix the labelling by requiring that, on the hyper-surface $t = 0$, r coincide with the radius

$$R(0, r) = r \quad (12)$$

This corresponds to the following choice of $t_c(r)$

$$t_c(r) = \frac{2}{(n+1)} \frac{r^{(n+1)/2}}{\sqrt{F}} \quad (13)$$

The central singularity occurs at $r = 0$, the corresponding time being $t = t_0(0) = 0$. We denote by $\rho(r)$ the initial density:

$$\rho(r) \equiv \epsilon(0, r) = \frac{nF'}{2r^n} \Rightarrow F(r) = \frac{2}{n} \int \rho(r) r^n dr \quad (14)$$

It can be seen from Eq. (10) that the density diverges faster in HD as compared to 4D. Thus given a regular initial surface, the time for the occurrence of the central shell-focusing singularity for the collapse developing from that surface is reduced as compared to the 4D case for the marginally bound collapse. The reason for this stems from the form of the mass function in Eq. (14). In a ball of radius 0 to r , for any given initial density profile $\rho(r)$, the total mass contained in the ball is greater than in the corresponding 4D case. In the 4D case, the mass function $F(r)$ involves the integral $\int \rho(r) r^2 dr$ [12], as compared to the factor r^n in the HD case. Hence, there is relatively more mass-energy collapsing in the space-time as compared to the 4D case, because of the assumed overall positivity of mass-energy (energy condition). This explains why the collapse is faster in the HD case.

Clearly, the time coordinate and radial coordinate are, respectively, in the ranges $-\infty < t < t_c(r)$ and $0 \leq r < \infty$.

3 Existence and Nature of Naked Singularity

In the context of Tolman-Bondi space-times, shell crossing singularities are defined by $R' = 0$ and they can be naked. It has been shown [5] that shell crossing singularities are gravitationally weak and hence such singularities cannot be considered seriously in the context of the CCC. On the other hand, central shell focusing singularities (characterized by $R = 0$ and $R' = 0$) are also naked and

gravitationally strong as well. Thus, unlike shell crossing singularities, shell focusing singularities do not admit any metric extension through them. Here we wish to investigate the similar situation in our HD space-time. Christodoulou [4] pointed out in the 4D case that the non-central singularities are not naked. Hence, we shall confine our discussion to the central shell focusing singularity. Eq. (11), by virtue of Eq. (13), leads to

$$R^{(n+1)/2} = r^{(n+1)/2} - \frac{(n+1)}{2} \sqrt{F} t \quad (15)$$

and the energy density becomes

$$\epsilon(t, r) = \frac{2n/(n+1)}{\left[t - \frac{2}{(n+1)} \frac{r^{(n+1)}}{\sqrt{F}} \right] \left[t - \frac{2\sqrt{F}}{F'} r^{(n-1)/2} \right]} \quad (16)$$

We are free to specify $F(r)$ and we consider a class of models which are non self-similar in general, and as a special case, the self-similar models can be constructed from them. In particular, we suppose that $F(r) = r^{(n-1)}\zeta(r)$ and $\zeta(0) = \zeta_0 > 0$ (finite). With this choice of $F(r)$, the density behaves as inversely proportional to the square of time at the centre, and $F(r) \propto r^{(n-1)}$ in the neighborhood of $r = 0$. For space-time to be self-similar, we require that $\zeta(r) = \text{const}$. This class of models for 4D space-time is discussed in refs. [8]. From Eq. (10) it is seen that the density at the centre ($r = 0$) behaves with time as $\epsilon = 2n/(n+1)t^2$. This means that the density is finite at any time $t = t_0 < 0$, but becomes singular at $t = 0$. Thus the singularity is interpreted as having arisen from the evolution of dust which had a finite density distribution in the past on an initial epoch.

We wish to investigate if the singularity, when the central shell with co-moving coordinate ($r = 0$) collapses to the centre at time $t = 0$, is naked. The singularity is naked iff there exists a null geodesic which emanates from the singularity. Let $K^a = dx^a/dk$ be the tangent vector to the radial null geodesic, where k is an affine parameter. Then we derive the following equations

$$\frac{dK^t}{dk} + \dot{R}' K^r K^t = 0 \quad (17)$$

$$\frac{dt}{dr} = \frac{K^t}{K^r} = R' \quad (18)$$

The last equation, upon using Eq. (9), turns out to be

$$\frac{dt}{dr} = \frac{r^{(n-1)/2} - \frac{F'}{\sqrt{F}}t}{\left[r^{(n+1)/2} - \frac{n+1}{2}\sqrt{F}\right]^{\frac{n-1}{n+2}}} \quad (19)$$

Clearly this differential equation becomes singular at $(t, r) = (0, 0)$. We now wish to put Eq. (19) in a form that will be more useful for subsequent calculations. To this end, we define two new functions $\eta = rF'/F$ and $P = R/r$. From Eq. (9), for $f = 0$, we have $\dot{R} = -\sqrt{F}/R^{(n-1)}$, and we can express F in terms of r by $F(r) = r^{(n-1)}\zeta(r)$. Eq. (19) can thus be re-written as

$$\frac{dt}{dr} = \left[\frac{t}{2r}\eta - \frac{1}{\sqrt{\zeta}} \right] \dot{R} \quad (20)$$

It can be seen that the functions $\eta(r)$ and $P(r, t)$ are well defined when the singularity is approached.

Let us define $X = t/r$ as usual. From the definitions of P and X , and Eq. (15), we can derive the following equation

$$X - \frac{2}{(n+1)\sqrt{\zeta}} = \frac{-P^{(n+1)/2}}{(n+1)\sqrt{\zeta}} \quad (21)$$

The nature (a naked singularity or a black hole) of the singularity can be characterized by the existence of radial null geodesics emerging from the singularity. The singularity is at least locally naked if there exist such geodesics, and if no such geodesics exist, it is a black hole. If the singularity is naked, then there exists a real and positive value of X_0 as a solution to the algebraic equation [12]

$$X_0 = \lim_{t \rightarrow 0} \lim_{r \rightarrow 0} X = \lim_{t \rightarrow 0} \lim_{r \rightarrow 0} \frac{t}{r} = \lim_{t \rightarrow 0} \lim_{r \rightarrow 0} \frac{dt}{dr} = R' \quad (22)$$

We insert Eq. (20) into Eq. (22) and use the results $\lim_{r \rightarrow 0} \eta = n - 1$ and $\lim_{r \rightarrow 0} \dot{R} = -\sqrt{\lambda}/Q_0^{(n-1)/2}$ to get

$$X_0 = \frac{1}{Q_0^{(n-1)/2}} \left[1 - \frac{n-1}{2} X_0 \sqrt{\zeta_0} \right] \quad (23)$$

where $Q(X) = P(X, 0)$. From Eq. (21) we can find Q_0 and then substituting Q_0 into Eq. (23), we get the algebraic equation

$$X_0 \left[1 - \frac{(n+1)}{2} \sqrt{\zeta_0} X_0 \right]^{(n-1)/(n+1)} + \frac{(n-1)}{2} \sqrt{\zeta_0} X_0 - 1 = 0 \quad (24)$$

which can be written as

$$X_0 \left[1 - \frac{1}{\Theta_{(n+2)}^0} X_0 \right]^{(n-1)/(n+1)} + \frac{(n-1)}{(n+1)\Theta_{(n+2)}^0} X_0 - 1 = 0 \quad (25)$$

where Θ_{n+2} is defined by $t_c(r) = \Theta_{(n+2)} r$ and $\Theta_{(n+2)}^0 = \Theta_{n+2}(0)$. From Eq. (21), it is clear that $0 < X < \Theta_{n+2}$, as P is a positive function. This algebraic equation governs the behaviour of the tangent near the singular points. The central shell focusing is at-least locally naked (for brevity we address it as naked throughout this paper) if Eq. (25) admits one or more positive roots subject to the constraint that $0 < X_0 < \Theta_0^{(n+2)}$. The values of the roots give the tangents of the escaping geodesics near the singularity. The smallest of value of X_0 , say X_0^s , corresponds to the earliest ray escaping from the singularity and is called the Cauchy horizon of the space-time and there is no solution in the region $X < X_0^s$. Thus in the absence of a positive root to Eq. (25), the central singularity is not naked because there are no outgoing future directed null geodesics emanating from the singularity.

To extract some information from our analysis we start from the standard 4D Tolman-Bondi case, which has been analyzed earlier by many authors (see for example, [8]). We already know what happens and we refer the reader to these papers for details. Setting $n = 2$, Eq. (25) simplifies to

$$X_0 \left[1 - \frac{3}{2} \sqrt{\zeta_0} X_0 \right]^{1/3} + \frac{1}{2} \sqrt{\zeta_0} X_0 - 1 = 0 \quad (26)$$

To facilitate comparison with Singh and Joshi [8], we introduce a new variable y defined as

$$y = \sqrt{\zeta_0} X_0 \quad (27)$$

After some rearrangement, equation (26) takes the form as in Joshi and Singh [8]:

$$y^3(y - \frac{2}{3}) - \gamma(y - 2)^3 = 0 \quad (28)$$

where $\gamma = \zeta^{(3/2)}/12$ and $y < 2/3$. It can be shown that Eq. (28) has positive roots, subject to the constraint that $y < 2/3$, if $\gamma < \gamma_1 = 26/3 - 5\sqrt{3} \approx 6.41626 \times 10^{-3}$ or equivalently if $\zeta_0 \leq \zeta_0^C = 0.1809$. For example if $\zeta_0 = 0.1808$, then the two roots of Eq. (26) are $y = 0.530448$, $y = 0.541188$ and the corresponding values of X_0 are shown in table-II below. Eq. (26) admits positive real roots for $\gamma < \gamma_2 = 26/3 + 5\sqrt{3} \approx 17.32$ also. The range $\gamma > \gamma_2$ however, is unphysical because it corresponds to imaginary X_0 [8]. Thus referring to our above discussion, collapse leads to a naked singularity for $\gamma < \gamma_1$ ($\zeta_0 \leq 0.1809$) and to a black hole otherwise. This agrees with the earlier results [8].

We now wish to investigate the changes introduced, at least qualitatively, in the above picture by the introduction of extra dimensions. To conserve space, we avoid repetition of the detailed analysis (being similar to 4D case) and rather summarize only the main results in the following tables:

Table I: The Variation of the ζ_0^C , Θ_{n+2}^0 and X_0 with $D = n + 2$

| n | D | ζ_0^C | Θ_{n+2}^0 | Equal tangents (X_0) |
|-----|-----|-----------------|------------------|--------------------------|
| 2 | 4 | 0.18091652297 | 1.56736 | 1.25992 |
| 3 | 5 | 0.090169943745 | 1.6651 | 1.27202 |
| 4 | 6 | 0.056372858735 | 1.68471 | 1.26751 |
| 5 | 7 | 0.039372532807 | 1.67989 | 1.25992 |
| 6 | 8 | 0.0182853577435 | 2.11291 | 1.45161 |

Table II: The two tangents for a $\zeta < \zeta_0^C$, in the different space-times.

| n | D | $\zeta_0 < \zeta_0^C$ | Two tangents (X_0) |
|-----|-----|-----------------------|------------------------|
| 2 | 4 | 0.1808 | 1.2476, 1.27277 |
| 3 | 5 | 0.0900 | 1.25, 1.29571 |
| 4 | 6 | 0.0563 | 1.24925, 1.28692 |
| 5 | 7 | 0.0393 | 1.23863, 1.28284 |
| 6 | 8 | 0.1820 | 1.40558, 1.50272 |

The quantities ζ_0 , Θ_{n+2} and X_0 depend on the dimension of the space-time. Thus it follows that the singularity is naked if $\zeta_0 \leq \zeta_0^C$ (of course the roots are subject to the constraint that $X_0 < \Theta_{n+2}$). On the other hand, if the inequality is reversed, no naked singularity forms and gravitational collapse of the dust collapse will result in a black hole. The quantity ζ_0^C can be called the critical parameter as at $\zeta_0^C = \zeta_0$, a transition occurs and the end state of collapse turns from a naked singularity to a black hole. It is interesting to note that ζ_0^C decreases as one introduces extra dimensions, whereas Θ_{n+2} increases. We have plotted a graph of critical parameter ζ_0^C against dimension D (see Figure I). Thus we can say that the formation of a black hole is facilitated with introduction of the extra dimensions or, in other words, the naked singularity spectrum gets continuously covered in higher dimensional space-times. Thus it appears that the singularity will be completely covered for very large dimensions of the space-time. Our results are in agreement with earlier work [15] on 5D spherical inhomogeneous dust collapse. A similar situation also occurs in HD radiation collapse [16].

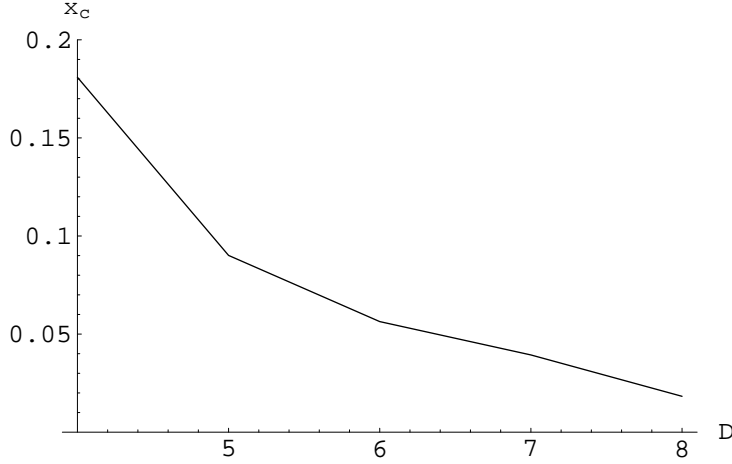


Figure 1: Plot of ζ_0^C versus D

3.1 Strength of Naked Singularity

Finally, we need to determine the curvature strength of the naked singularity which is an important aspect of a singularity [17]. There has been an attempt to relate the strength of a singularity to its stability [10]. A singularity is gravitationally strong or simply strong if it destroys by crushing or stretching any object which fall into it and weak if no object which falls into the singularity is destroyed in this way. It is widely believed that a space-time does not admit an extension through a singularity if it is a strong curvature singularity in the sense of Tipler [18]. Clarke and Królak [19] have shown that a sufficient condition for a strong curvature singularity as defined by Tipler [18] is that for at least one non-space like geodesic with affine parameter k , in the limiting approach to the singularity, we must have

$$\lim_{k \rightarrow 0} k^2 \psi = \lim_{k \rightarrow 0} k^2 R_{ab} K^a K^b > 0 \quad (29)$$

where R_{ab} is the Ricci tensor. Our purpose here is to investigate the above condition along future directed radial null geodesics which emanate from the naked singularity. Now (29), with the help of Eq. (3) and using $P = R/r$, can

be expressed as

$$\lim_{k \rightarrow 0} k^2 \psi = \lim_{k \rightarrow 0} \frac{nF'}{2r^{(n-2)}P^{(n-2)}R'} \left[\frac{kK^t}{R} \right]^2 \quad (30)$$

In order to evaluate the above limit, we note that the tangent K^t blows up in the limiting approach to the naked singularity. Using L'Hospital's rule, we find that

$$\lim_{k \rightarrow 0} k^2 \psi = \lim_{k \rightarrow 0} \frac{k}{R \frac{1}{K^t}} = \lim_{k \rightarrow 0} \frac{1}{\frac{R\dot{R}'}{R'} + \dot{R} + 1} \quad (31)$$

The quantity \dot{R}' , from Eq. (9), can be calculated as:

$$\dot{R}' = -\frac{1}{R} \left[\frac{\eta\sqrt{\lambda}}{2P^{\frac{(n-1)}{2}-1}} + \frac{n-1}{2}\dot{R}R' \right] \quad (32)$$

Now Eq. (29), because of Eqs. (30), (31) and (32), and using the fact that $F/r^{(n-1)}$ and R' tend to finite values $\lambda_0(n-1)$ and X_0 respectively, yields

$$\lim_{k \rightarrow 0} k^2 \psi = \frac{n\lambda_0(n-1)}{\left[[(n-3)X_0 - (n-1)Q_0]\sqrt{\lambda_0} + 2X_0Q_0^{(n-1)/2} \right]^2} > 0 \quad (33)$$

Thus along radial null geodesics coming out from singularity $\lim_{k \rightarrow 0} k^2 \psi$ is finite and hence the strong curvature condition is satisfied.

4 Concluding remarks

The Tolman-Bondi metric in the 4D case is extensively used for studying the formation of naked singularities in spherical gravitational collapse. Indeed, both analytical [8]-[9] and numerical results [3] in dust indicate the critical behaviour governing the formation of black holes or naked singularities. For further progress towards an understanding of spherical dust collapse, from the viewpoint of the CCC, one would like to know the effect of extra dimensions on the existence of a naked singularity. The relevant questions would be, for instance, do such solutions remain naked with the introduction of extra dimensions? Do they always become covered? Does the nature of the singularity change? Our analysis shows that

none of the above hold. Indeed, the gravitational collapse of inhomogeneous dust in higher dimensional space-time leads to a strong curvature naked singularity. Thus extra dimensions cannot completely cover the naked singularity nor can it affect the nature of the singularity. However, we showed that the effect of extra dimensions appears to be a shrinking of the naked singularity initial data space (of 4D) or an enlargement of the black hole initial data space. Thus one can say that the naked singularity spectrum in 4D case gets covered with the introduction of extra dimensions in the space-time or one can say that extra dimensions of the space-time facilitate the formation of black holes in comparison to naked singularities.

This generalizes the previous studies of spherical gravitational collapse in 4D to HD space-times and when $n = 2$, one recovers the conventional 4D Tolman-Bondi models. Also for $n = 3$, our result reduces to those obtained previously for the 5D case [15]. The formation of these naked singularities violates the strong CCC. We do not claim any particular physical significance to the HD metric considered. Nevertheless we think that the results obtained here have some interest in the sense that they do offer the opportunity to explore properties associated with naked singularities which may be crucial in our understanding of this important problem. Finally, the result obtained would also be relevant in the context of superstring theory which is often said to be next "theory of everything", and for an interpretation of how critical behaviour depends on the dimensionality of the space-time.

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